Distributed Estimation Fusion Under Unknown Cross-Correlation: An Analytic Center Approach

Yimin Wang
School of Electronic
and Information Engineering
Xi'an Jiaotong University
Xi'an, Shaanxi 710049, P. R. China
eric.wang.xjtu@gmail.com

X. Rong Li
Department of Electrical
Engineering
University of New Orleans
New Orleans, LA 70148, U.S.A.
xli@uno.edu

***Abstract*—We develop an analytic center approach to distributed estimation fusion when the cross-correlation of errors between local estimates is unknown. Based on a set-theoretic formulation of the problem, we seek an estimate that maximizes the complementary squared Mahalanobis "distance" between the local and the desired estimates in a logarithmic average form, and the optimal value turns out to be the analytic center. For our problem, we then prove that the analytic center is a convex combination of the local estimates. As such, our proposed analytic center covariance intersection (AC-CI) algorithm could be regarded as the covariance intersection (CI) algorithm with respect to a set-theoretic optimization criteria.**

***Keywords*: Distributed fusion, decentralized network, set-theoretic estimation, analytic center, covariance intersection, convex combination.**

I. Introduction

Estimation fusion is the problem of how to best utilize useful information contained in multiple sets of data for the purpose of estimating a quantity [1]. There are two basic fusion architectures, centralized and distributed, depending on whether raw data are used directly for fusion or not. In centralized fusion, all sensors send their raw data to a fusion center, which can produce globally optimal estimates. In the standard distributed fusion, each local sensor processes its own raw data and sends the local estimate to other sensors or a fusion center to obtain global estimates. Two basic distributed fusion architectures are well known: hierarchical and fully distributed [6], and sometimes the latter is referred to as decentralized networks. Generally, distributed fusion may have a less communicational burden and higher survivability, and is more flexible and reliable.

However, distributed fusion also has challenging problems not present in centralized fusion. One of the major issues is dealing with the cross-correlation of errors of local estimates due to common process noise, prior information, correlation among measurement noises across sensors, etc. Most existing distributed fusion algorithms (see, e.g., [1]–[8]) accounting for this problem try to exploit correlation in estimates.

Consider a decentralized network of M nodes whose topology is completely arbitrary (it might include loops) and can change dynamically. Each node has information only about its local topology (e.g., the number of nodes with which it directly communicates and the type of data sent across each communication link) [11]. In this case, the calculation of the cross-correlation of local estimation errors is quite involved and not practical. [9]–[11] proposed the Covariance Intersection (CI) algorithm under unknown cross-correlation to bypass this problem. The CI algorithm provides a useful convex combination of the local estimates with weights in the information space, and the fused mean squared error (MSE) matrix by the CI algorithm is guaranteed to be a bound on the actual MSE matrix. In [10]–[11], the weight coefficients were obtained by minimizing the trace or determinant of the fused MSE matrix, which is called the general CI (GCI) algorithm in this paper. [13] and [14] also presented fast CI algorithms whose weight coefficients are determinted approximately. Besides the determinant minimization criterion, there are alternative information-theoretic and set-theoretic optimization criteria:

- Information-theoretic optimization criteria. [12] presented an information-theoretic justification for the CI algorithm and a criterion of minimizating Chernoff information, which is computationally complex. [15] proposed a fast CI algorithm (IT-FCI) in a closed form based on an information-theoretic criterion.
- Set-theoretic optimization criteria. [16] first provided a set-theoretic interpretation of the CI algorithm. Inspired by it, [17] adopted the theory of set-theoretic estimation (see, e.g., [19], [20] and the survey paper [21]) to formulate the problem of distributed estimation fusion under unknown cross-correlation, and presented a relaxed Chebyshev center covariance intersection (RCC-CI) algorithm.

In this paper, we follow the set-theoretic formulation which provides a natural way to represent the information from the

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local estimates with unknown correlation [37]. Each piece of information is associated with a set (e.g., an ellipsoid) in the solution space, and the acceptable solutions would lie in the intersection of these sets, called the feasible set (FS), whose central representative could be considered as a final fused estimate [17]. In [16], the center of bounding ellipsoid for FS was chosen to be the fused estimate. Another popular way is the Chebyshev center (see, e.g., [17], [22]–[24]), which minimizes the worst-case estimation error over the FS. Inspired by [36], here we seek an estimate that maximizes the complementary squared Mahalanobis "distance" between the local estimate and the desired estimate in a logarithmic average form, and the optimal value turns out to be the so-called analytic center (AC) (see, e.g., [27]–[34]). We then prove that the analytic center is a convex combination of the local estimates. As such, like the CI algorithm, the proposed analytic center covariance intersection (AC-CI) algorithm can provide a fused MSE matrix that is guaranteed to be a bound on the actual conditional MSE matrix and can also be regarded as a CI algorithm with respect to a set-theoretic optimization criterion. The proposed set-theoretic approach AC-CI as well as RCC-CI [17] is characterized by that both the value and uncertainty of local estimates are taken into account to obtian the weight coefficients, whereas the determinant optimization criterion only focus on the uncertainty.

The structure of this paper is as follows. In Section II, we review the covariance intersection algorithm (due to sapce limitation, the IT-FCI, the approximate fast CI algorithms and the fusion algorithms for incoherent local estimates ([17], [18]) are not covered). We then briefly introduce set-theoretic estimation in Section III and present an AC-CI algorithm. Illustrative simulation examples are given in Section IV. Section V provides conclusions.

II. Covariance Intersection

Consider a fully distributed system or a decentralized network with M nodes whose connection topology is completely arbitrary. Each node has information only about its local topology. Denote by $x \in \mathbb{R}^k$ the estimand (i.e., the quantity to be estimated). Suppose N available local estimates are to be fused at a fusion node. Denote by $\hat{x}_i$ the ith available local estimate and by P_i the corresponding MSE matrix. The local estimate $\hat{x}_i$ could be biased or unbiased. Let $P_i^* = E\left[\tilde{x}_i\tilde{x}_i^T\right]$ be the actual MSE matrix of its associated estimation error $\tilde{x}_i = x - \hat{x}_i$, and $P_{ij}^* = E\left[\tilde{x}_i\tilde{x}_j^T\right]$ be the actual cross-correlation matrix between estimation errors $\tilde{x}_i$ and $\tilde{x}_j$.

The problem considered in this paper is how to obtain global estimate $\hat{x}$ and the corresponding MSE matrix P using all available local estimates at the fusion node. Since calculating the cross-correlation of local estimation errors in a such system is quite involved and impractical, we assume that the cross-correlation is *unknown*, which means that the only available information is the local estimates and their corresponding MSE matrices $(\hat{x}_i, P_i)$—no knowledge on the cross-correlation between the local estimates is available.

In this paper, an estimate $\hat{x}_i$ is said to be *conservative* (or pessimistic definite [25]) if

$$P_i - P_i^* \geq 0 \tag{1}$$

where P_i^* is the actual MSE matrix of $\hat{x}_i$. Let an n-sigma P-ellipsoid be the locus of points $\left\{p | p^T P^{-1} p = n\right\}$. A conservative estimate $\hat{x}_i$ means that P_i-ellipsoid covers the P_i^*-ellipsoid.

Suppose two estimates are to be fused, $\hat{x}_i,\ i = 1, 2$, with MSE matrix P_i. The CI algorithm, which takes a convex combination of $\hat{x}_1$ and $\hat{x}_2$ along with their MSE matrices, is given by

$$\hat{x}_{CI} = P_{CI}\left(\omega_1 P_1^{-1}\hat{x}_1 + \omega_2 P_2^{-1}\hat{x}_2\right) \tag{2}$$

$$P_{CI} = \left(\omega_1 P_1^{-1} + \omega_2 P_2^{-1}\right)^{-1} \tag{3}$$

$$\omega_1 + \omega_2 = 1 \tag{4}$$

where weight coefficients $\omega_1,\ \omega_2 \in [0, 1]$.

CI provides a nice mechanism for distributed fusion under unknown cross-correlation: First, the global estimate is obtained by a convex combination, and is unbiased if all the local estimates are unbiased. Second, the CI yields a conservative estimate for any value of $\omega_1,\ \omega_2$ and P_{ij}^* providing that all the local estimates are conservative. That is, $P_{CI} - P_{CI}^* \geq 0$, where

$$\begin{aligned} P_{CI}^* = P_{CI} &\left[\omega_1^2 P_1^{-1} P_1^* P_1^{-1} + \omega_1\omega_2 P_1^{-1} P_{12}^* P_2^{-1}\right. \\ &\left. + \omega_1\omega_2 P_2^{-1} P_{21}^* P_1^{-1} + \omega_2^2 P_2^{-1} P_2^* P_2^{-1}\right] P_{CI} \end{aligned} \tag{5}$$

is the actual MSE matrix of $\hat{x}_{CI}$. Note that P_{CI}^* could be conditional or unconditional MSE matrix depending on whether the weight coefficients $\omega_1,\ \omega_2$ are independent of $\hat{x}_i$.

If the weight coefficients ω_1 and ω_2 are chosen to minimize $\det(P_{CI})$, it is called the *general CI* (***GCI***) algorithm (see, e.g., [10], [11]) in this paper. Then, under the determinant minimization criterion, the fused estimate is denoted as $(\hat{x}_{GCI}, P_{GCI})$.

CI can be generalized to an arbitrary number of $N \geq 2$ updates [11]:

$$\hat{x}_{CI} = P_{CI}\sum_{i=1}^{N}\omega_i P_i^{-1}\hat{x}_i \tag{6}$$

$$P_{CI} = \left(\sum_{i=1}^{N}\omega_i P_i^{-1}\right)^{-1} \tag{7}$$

$$\sum_{i=1}^{N}\omega_i = 1 \tag{8}$$

III. Analytic Center Approach

A. Set-Theoretic Estimation

Set-theoretic estimation was pioneered by Schweppe [19], [20]. In this framework, each piece of information is represented by a set in the solution space and the intersection of such sets constitutes all possible solutions, i.e., the feasible set (FS). In [21], set-theoretic estimation was defined as "an estimation framework in which consistency of a solution with the observed data and all *a priori* knowledge serves as the criterion of acceptability," and a formal definition of the set-theoretic formulation was given as follows.

Considering the general estimation problem where the estimand x belongs to a solution space $\mathbb{R}^k$ and $\hat{x} \in \mathbb{R}^k$ is a proposed estimate. A piece of information is represented by a mapping $\Psi_i : \mathbb{R}^k \to [0, 1]$, called fuzzy proposition, which assigns to every point $\hat{x} \in \mathbb{R}^k$ a grade of "consistency" $\Psi_i(\hat{x})$ $(i = 1, 2, \ldots, n)$. Let ψ_i be the real numbers in $[0, 1]$ representing the strength of the beliefs that the estimand satisfies these propositions. Then, a family $\{S_i\}$ of subsets of $\mathbb{R}^k$ can be constructed as

$$S_i = \left\{\hat{x} \in \mathbb{R}^k | \Psi_i(\hat{x}) \geq \psi_i\right\} \tag{9}$$

Each S_i is called a property set. Thus, S_i is the set of all estimates that are consistent with the information carried by Ψ_i at level ψ_i. The pair $\left(\mathbb{R}^k, \{S_i\}\right)$ will be called a set-theoretic formulation of the problem. The subset of $\mathbb{R}^k$ "consistent" with all available information is the FS or solution set:

$$\Omega = \bigcap_i S_i \tag{10}$$

Any point in Ω is called a set-theoretic estimate.

B. Analytic Center

Within the framework of set-theoretic estimation, each available local estimate $(\hat{x}_i, P_i)$ at the fusion node could be represented by an ellipsoidal approximation as the property set [19]:

$$\mathcal{E}_i = \left\{x | (x - \hat{x}_i)^T P_i^{-1} (x - \hat{x}_i) < 1\right\} \tag{11}$$

Then the FS is given by the intersection of the N ellipsoids:

$$\Omega = \left\{x | (x - \hat{x}_i)^T P_i^{-1} (x - \hat{x}_i) < 1,\ 1 \leq i \leq N\right\} \tag{12}$$

The remaining problem is to choose a central representative $\hat{x} \in \Omega$ as the final estimate. Assuming Ω is not empty, a popular way is to find the Chebyshev center of Ω (see, e.g., [17], [22]–[24]), which is equivalent to minimizing the worst-case error over all feasible estimates:

$$\min_{\hat{x}} \max_{x \in S} \|x - \hat{x}\|^2 \tag{13}$$

where $\|x\|$ stands for the Euclidean norm of the vector x.

Let

$$d_{P_i}^2(\hat{x}, \hat{x}_i) = (\hat{x} - \hat{x}_i)^T P_i^{-1} (\hat{x} - \hat{x}_i) \tag{14}$$

denote the squared Mahalanobis distance between the desired estimate $\hat{x}$ and the local estimate $\hat{x}_i$ [26]. The smaller $d_{P_i}^2(\hat{x}, \hat{x}_i)$ is, the closer $\hat{x}$ is to $\hat{x}_i$ in the ith Mahalanobis space (MS) with $d_{P_i}(\hat{x}, \hat{x}_i)$ as a metric on $\mathbb{R}^k$. We refer to

$$f_{P_i}(\hat{x}, \hat{x}_i) = 1 - d_{P_i}^2(\hat{x}, \hat{x}_i) \tag{15}$$

as the complementary squared Mahalanobis "distance" between $\hat{x}$ and $\hat{x}_i$ in the ith MS. Then Ω in Eq. (12) can be represented as

$$\Omega = \{x | f_{P_i}(x, \hat{x}_i) > 0,\ 1 \leq i \leq N\} \tag{16}$$

In order to find a central representative in the FS, we propose to maximize the complementary squared Mahalanobis "distance" in a logarithmic average form:

$$\max_{x \in \Omega} \prod_{i=1}^{N} f_{P_i}(x, \hat{x}_i) \quad \text{or} \quad \max_{x \in \Omega} \sum_{i=1}^{N} \ln f_{P_i}(x, \hat{x}_i)$$

It turns out that the optimal value of x:

$$\begin{aligned} \hat{x}_{AC} &= \arg\max_{x \in \Omega} \sum_{i=1}^{N} \ln f_{P_i}(x, \hat{x}_i) \\ &= \arg\min_{x \in \Omega} \phi(x) \end{aligned} \tag{17}$$

is the so-called *analytic center* of Ω, where

$$\phi(x) = \begin{cases} -\sum_{i=1}^{N} \ln f_{P_i}(x, \hat{x}_i) & x \in \Omega \\ \infty & x \notin \Omega \end{cases} \tag{18}$$

is referred to as the *potential function* of Ω [32]. The analytic center was first introduced in [27] and has been widely studied in the optimization literature (see, e.g., [28]–[34]).

Here we convert each convex quadratic constraint in Eq. (16) to a linear matrix inequality (LMI):

$$G_{blk_i}(x) = \begin{bmatrix} I_{k \times k} & U_i (x - \hat{x}_i) \\ (x - \hat{x}_i)^T U_i & 1 \end{bmatrix} > 0 \tag{19}$$

where $U_i = P_i^{-1/2}$. Then the LMI

$$G(x) = \operatorname{diag}(G_{blk_1}(x), \cdots, G_{blk_N}(x)) > 0 \tag{20}$$

in the variable x has the interior of Ω as the FS [28]. Then the function $\phi(x)$ has an equivalent form (see, e.g., [29], [30]):

$$\phi(x) = \begin{cases} \ln \det G(x)^{-1} & x \in \Omega \\ \infty & x \notin \Omega \end{cases} \tag{21}$$

which is called a *barrier function* for Ω.

Suppose that the FS Ω is nonempty. Since the objective function $\phi(x)$ is strictly convex on Ω, the analytic center $\hat{x}_{AC}$ is the *unique* minimizer of the optimization problem (17). The $\hat{x}_{AC}$ is also referred to as the analytic center of the LMI $G(x) > 0$. This analytic center problem is a special case of maxdet-problem [31] and can be solved by using efficient determinant optimization routines [35].

Remark 1: Although we do not assume that the local estimate $(\hat{x}_i, P_i)$ is Gaussian, the ellipsoidal approximation shown in Eq. (11) is most suitable for the Gaussian case.

Remark 2: When the cross-correlation of errors between local estimates is unknown, we can crudely assume that the errors between local estimates are uncorrelated and then the weighted least squares (WLS) estimate is given by

$$\hat{x}_{WLS} = \arg\min_{x} \sum_{i=1}^{N} d_{P_i}^2(x, \hat{x}_i) \tag{22}$$

based on a *arithmetic average* criterion, which minimizes the squared Mahalanobis distance on average. It is also known as a simple convex combination [16]. However, our proposed criterion, which maximizes the product of complementary

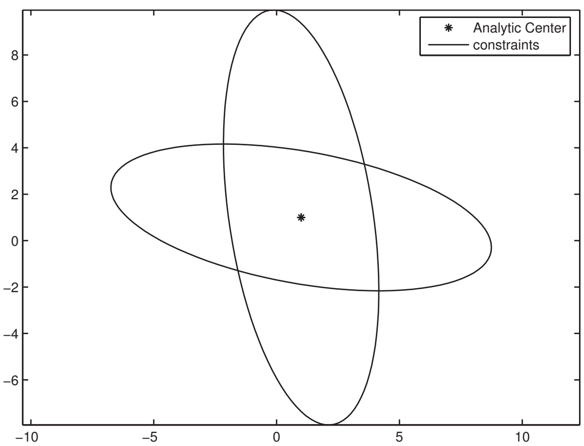


Fig. 1. Analytic center without redundant constraints

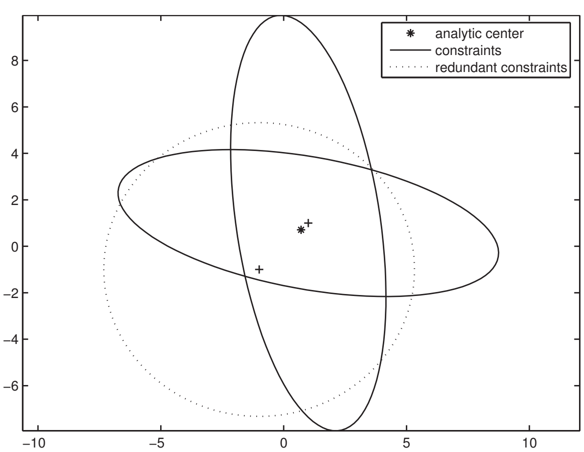


Fig. 2. Analytic center with redundant constraints

squared Mahalanobis "distances" can be viewed as a *geometric average* one.

Remark 3: The analytic center depends on the algebraic representation (e.g., LMI $G(x) > 0$ given in (20)), not the (strict) FS (see, e.g., [28], [29] and [32]). Different representations of the same FS may have different analytic centers. To illustrate, Fig. 1 shows the case of identical local estimates, for which the analytic center coincides with the local estimates. In Fig. 2, we add a redundant constraint. Note that although the FS in these two figures are identical, the two analytic centers are different because the redundant constraint provides additional information and moves the analytic center off to one side.

C. Convex Combination Form

In Theorem 1 below, we show that the analytic center $\hat{x}_{AC}$ can be represented as a convex combination of the local estimates. Given $\hat{x}_{AC}$ of (17), the weight coefficients can be determined analytically.

Theorem 1: The analytic center $\hat{x}_{AC}$ of (17) of the LMI (20) can be represented as

$$\hat{x}_{AC} = \left(\sum_{i=1}^{N} \omega_i P_i^{-1}\right)^{-1} \sum_{i=1}^{N} \omega_i P_i^{-1} \hat{x}_i \tag{23}$$

where

$$\omega_i = \frac{\prod_{j=1,j\neq i}^{N} f_{P_j}\left(\hat{x}_{AC}, \hat{x}_j\right)}{\sum_{i=1}^{N}\prod_{j=1,j\neq i}^{N} f_{P_j}\left(\hat{x}_{AC}, \hat{x}_j\right)}. \tag{24}$$

Proof: Since the analytic center $\hat{x}_{AC}$ is the *unique* minimizer of the optimization problem (17) and

$$\hat{x}_{AC} = \arg\min_{x\in\Omega} \phi(x) = \arg\max_{x\in\Omega} \psi(x)$$

where

$$\psi(x) = \prod_{i=1}^{N} f_{P_i}\left(x, \hat{x}_i\right) \tag{25}$$

we have

$$\begin{aligned} 0 &= \left.\frac{\partial \psi(x)}{\partial x}\right|_{x=\hat{x}_{AC}} \\ &= \sum_{i=1}^{N} \prod_{j=1,j\neq i}^{N} f_{P_j}\left(\hat{x}_{AC}, \hat{x}_j\right) f'_{P_i}\left(\hat{x}_{AC}, \hat{x}_i\right) \\ &= -2\sum_{i=1}^{N} \prod_{j=1,j\neq i}^{N} f_{P_j}\left(\hat{x}_{AC}, \hat{x}_j\right) P_i^{-1}\left(\hat{x}_{AC} - \hat{x}_i\right) \end{aligned} \tag{26}$$

that is,

$$\begin{aligned} &\left(\sum_{i=1}^{N} \prod_{j=1,j\neq i}^{N} f_{P_j}\left(\hat{x}_{AC}, \hat{x}_j\right) P_i^{-1}\right)\hat{x}_{AC} \\ &= \sum_{i=1}^{N} \prod_{j=1,j\neq i}^{N} f_{P_j}\left(\hat{x}_{AC}, \hat{x}_j\right) P_i^{-1} \hat{x}_i \end{aligned} \tag{27}$$

Substituting Eq. (24) into (27) yields

$$\hat{x}_{AC} = \left(\sum_{i=1}^{N} \omega_i P_i^{-1}\right)^{-1} \sum_{i=1}^{N} \omega_i P_i^{-1} \hat{x}_i \tag{28}$$

■

D. Analytic Center Covariance Intersection Algorithm

According to Theorem 1, the analytic center estimate has the same form as the CI estimate except that the weight coefficients ω_i are determined based on a different criterion. Like the CI algorithm, here we propose an analytic center covariance intersection (AC-CI) algorithm for the fusion problem, and the fused estimate $\hat{x}_{AC}$ and the corresponding MSE matrix P_{AC} are given by

$$\hat{x}_{AC} = \arg\max_{x\in\Omega} \phi(x) \tag{29}$$

$$\omega_i = \frac{\prod_{j=1,j\neq i}^{N} f_{P_j}\left(\hat{x}_{AC}, \hat{x}_j\right)}{\sum_{i=1}^{N}\prod_{j=1,j\neq i}^{N} f_{P_j}\left(\hat{x}_{AC}, \hat{x}_j\right)} \tag{30}$$

$$P_{AC} = \left(\sum_{i=1}^{N} \omega_i P_i^{-1}\right)^{-1} \tag{31}$$

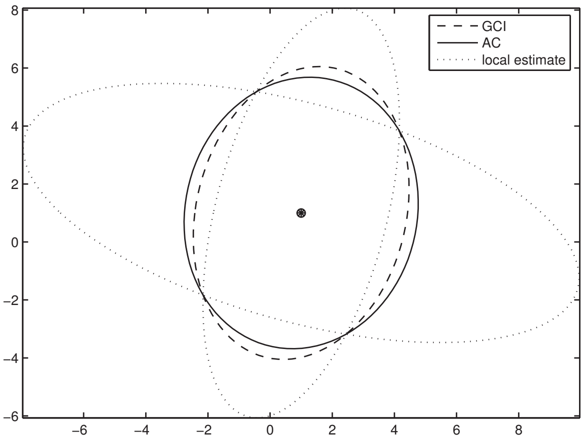


Fig. 3. Equal means

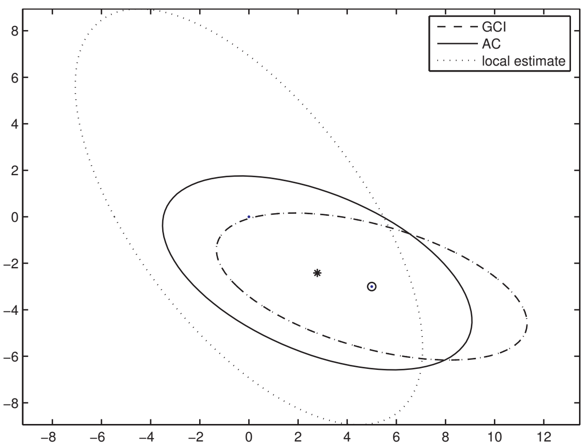


Fig. 4. Unequal means: GCI overlaps with a local estimate.

That is, first calculate $\hat{x}_{AC}$ by the proposed optimization method; then determine the weight coefficients ω_i; finally, obtain P_{AC}. If all local estimates are conservative, then $(\hat{x}_{AC}, P_{AC})$ is guaranteed to be conservative, meaning that $P_{AC} - P_{AC}^* \geq 0$, where P_{AC}^* is the actual MSE matrix of $\hat{x}_{AC}$ given ω_i. Thus, the AC-CI algorithm can be viewed as a CI algorithm with respect to a set-theoretic optimization criterion.

Figs. 3 and 4 compare the fused estimates of each pair of estimates calculated by GCI and AC-CI, respectively. In Fig. 3, the means of local estimates are equal. In Fig. 4, the means are not equal and $P_1 < P_2$. It can be shown that $(\hat{x}_{GCI}, P_{GCI})$ and $(\hat{x}_1, P_1)$ have the same ellipse. So, AC-CI seems more appealing.

Remark 1: For the determinant minimization criterion (i.e., GCI), if $P_i < P_j$, $i \neq j$, the fused estimate will be $(\hat{x}_i, P_i)$, even if the two estimates are far from each other. Since the determinant minimization criterion, which focuses on the uncertainty of the estimate, does not utilize the values of $\hat{x}_i$, the optimal $\omega_1, \ldots, \omega_N$ is independent of the values of $\hat{x}_i$, so the fused MSE matrix P_{GCI} is guaranteed to be a bound on the actual unconditional MSE matrix. As P_i increases from below P_j to above P_j, the GCI fused estimate will jump from $(\hat{x}_i, P_i)$ to $(\hat{x}_j, P_j)$, which is highly undesirable. The proposed set-theoretic criterion is to find the analytic center of an LMI, and both the value and uncertainty of estimates are taken into account. For AC-CI, since the weight coefficients ω_i depend on the value of local estimates $\hat{x}_i$, the fused MSE matrix P_{AC} is guaranteed to be conservative with respect to actual conditional MSE matrix given $\omega_1, \ldots, \omega_N$.

Remark 2: When the local estimates are *incoherent* [17], we could combine the proposed criterion with the fault-tolerant convex combination fusion approach (see, e.g., [15] and [17]) to obtain a more robust fused estimate. In a practical system, the property sets do not always intersect, and if FS is empty, we can change the level ψ_i in Eq. (9) to obtain larger property sets (e.g., use n-sigma ellipsoids, $n > 1$).

Remark 3: In [36], an analytic center approach for bounded error parameter estimation is presented, which is to find an estimate $\hat{\theta}$ of a deterministic parameter vector θ from noisy observation

$$y_i = \varphi_i \theta + w_i \tag{32}$$

where φ_i is a known model matrix and w_i is the noise vector bounded by $|w_i| < \epsilon$, $\epsilon > 0$. Thus, the FS for this problem is determined by linear inequalities:

$$\Theta = \bigcap_i \{\theta \,|{-\epsilon} < y_i - \varphi_i \theta < \epsilon\} \tag{33}$$

Focusing on the state estimation fusion, our proposed AC-CI algorithm, whose FS Ω shown in (16) is represented by quadratic constraints, is not limited to parameter estimation fusion and can provide a conservative MSE matrix. We can also use the *unified linear data model* [1] to formulate the fusion problem: the local estimate can be viewed as an observation of the estimand using the following identity:

$$\hat{x}_i = x + (\hat{x}_i - x) = x + v_i \tag{34}$$

where $v_i = -\tilde{x}_i$ is the pseudo measurement noise. When the cross-correlation of errors between local estimates is unknown, the only available information is the local estimates and their corresponding MSE matrices, $(\hat{x}_i, P_i)$. We further assume that there is a bound on the pseudo measurement noise based on the set-theoretic estimation, i.e.,

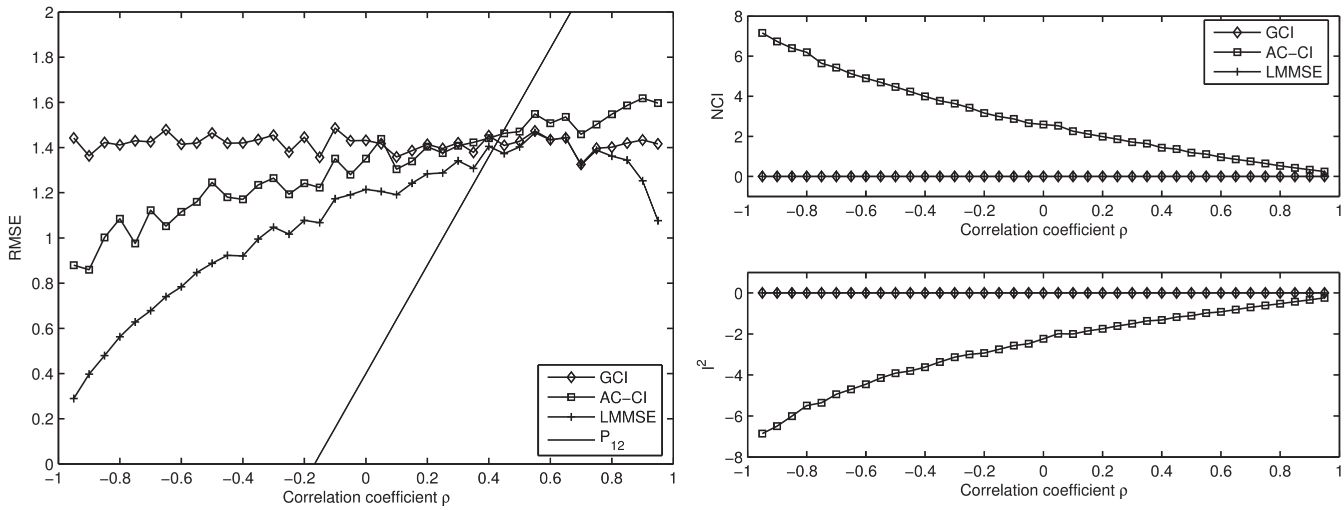
$$\|v_i\|^2_{P_i^{-1}} < 1 \tag{35}$$

where $\|\cdot\|_{P_i^{-1}} = \left(x^T P_i^{-1} x\right)^{1/2}$. We can obtain the same FS Ω as shown in (16) and the fused estimate can be solved by the AC-CI algorithm. As such, when the unified linear data model formulation is adopted, the AC algorithm provides an alternative set-theoretic way and can be viewed as a special bounded error estimation.

IV. Simulation and Comparison

A. Static Case

To simplify the evaluation of performance of the AC-CI and GCI [11] algorithms, we assume a scalar static Gaussian estimand $x \sim \mathcal{N}\left(\bar{x}, \bar{P}\right)$ with two local estimates [16] $(\hat{x}_i, P_i)$,



(a) RMSE (b) NCI and I^2

Fig. 5. Static Case: AC-CI, GCI and BLUE

$i = 1, 2$, which are the linear minimum mean-squared error (LMMSE) estimates with prior for the observation equation

$$z_i = x + v_i \tag{36}$$

where v_i is zero mean Gaussian noise with

$$\begin{aligned} \text{cov}(v_i, v_i) &= R_i \\ \text{cov}(v_i, v_j) &= R_{ij},\ i \neq j \end{aligned}$$

and

$$\begin{aligned} \hat{x}_i &= \bar{x} + \bar{P}\left(\bar{P} + R_i\right)^{-1}(z_i - \bar{z}_i) \\ P_i &= \left(\bar{P}^{-1} + R_i^{-1}\right)^{-1} \end{aligned}$$

The cross-correlation between local estimation errors can be calculated analytically as

$$P_{ij} = P_i \bar{P}^{-1} P_j^T + K_i R_{ij} K_j^T \tag{37}$$

where

$$K_i = \bar{P}\left(\bar{P} + R_i\right)^{-1}$$

Given correlation coefficient $-1 < \rho < 1$, the cross-covariance of noise is given by

$$R_{ij} = \rho\sqrt{R_i R_j}$$

For this example, we also provide the LMMSE fusion [1] as a baseline. We vary ρ and compute the root mean squared error (RMSE), noncredibility indices (NCI) and inclination indicators (I^2) [25] for each algorithm from 1000 Monte Carlo runs in the following cases (see, Fig. 5):

$$\left(\bar{x}, \bar{P}\right) = (1, 20), R_1 = 2.22, R_2 = 5, P_1 = 2, P_2 = 4$$

As can be seen form Fig. 5(a) for RMSE, since $P_1 < P_2$, GCI ends up with the LMMSE estimate $\hat{x}_1$ and drops another LMMSE estimate $\hat{x}_2$. AC-CI performs well when P_{12} is small, especially when it is negative. This is understandable: For this scalar case, given $(\hat{x}_i, P_i)$ according to Eq. (11), each piece of information could be represented by a line segment:

$$\mathcal{E}_i = \left\{ p \,\middle|\, \hat{x}_i - \sqrt{P_i} < p < \hat{x}_i + \sqrt{P_i} \right\}$$

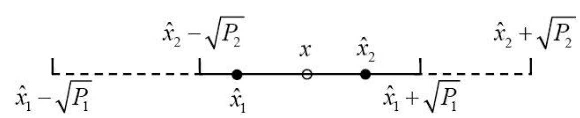
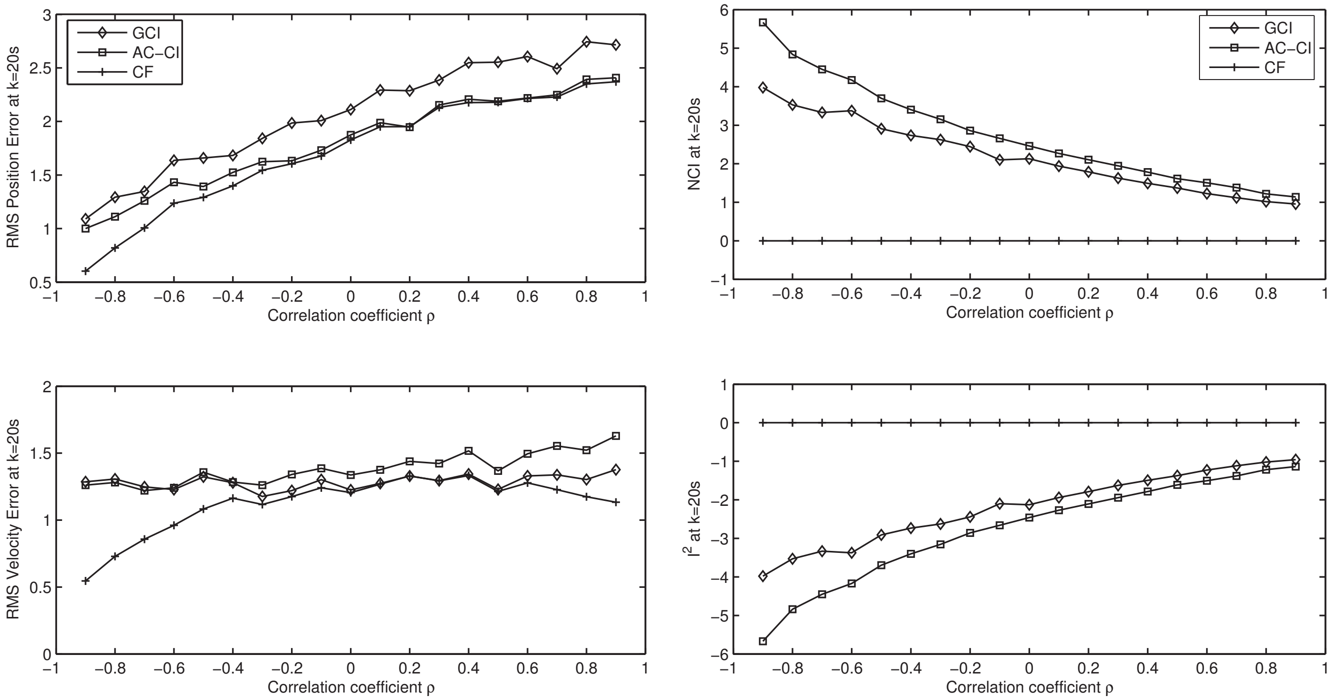


Fig. 6. Scalar case

As shown in Fig. 6, the solid line segment is the overlapped part of $\mathcal{E}_1$ and $\mathcal{E}_2$ as the FS, the dashed line segments are the unoverlapped parts. If the local estimates have a negative correlation, which means that the estimates deviate from the estimand in different directions, the estimand x would lie between $\hat{x}_1$ and $\hat{x}_2$ statistically. Then in most cases, the estimand would lie in the FS, $x \in \bigcap_i \mathcal{E}_i$, so the set-theoretic estimate AC would be a proper estimate. However, if the local estimates have a positive correlation, the estimand x would be larger or smaller than $\hat{x}_1$ and $\hat{x}_2$ statistically. For example, the estimand lies in the dashed line segments, $x \notin \bigcap_i \mathcal{E}_i$, and in this case the set-theoretic formulation is not good. As discussed above, AC-CI which considers both the value and uncertainty of the local estimates can make rational use of the correlation information when the correlation is small, especially negative.

As shown in Fig. 5(b) for NCI and I^2, since GCI estimate is just the LMMSE $(\hat{x}_1, P_1)$, GCI is credible. For AC-CI, the provided MSE matrix, which is the convex combination of P_1^{-1} and P_2^{-1}, lies in $[P_1, P_2]$. When P_{12} is small, although AC-CI have smaller RMSE than GCI, they both are pessimistic, since their I^2 are negative and about equal to NCI in magnitude [25].



(a) RMSE

(b) NCI and I^2

Fig. 7. Dynamic Case: AC-CI, GCI and BLUE

B. Dynamic Case

Consider a constant-velocity moving target in one dimension with a total time span of 20 seconds. Its dynamic model is assumed to be

$$x_{k+1} = Fx_k + Gw_k \tag{38}$$

where

$$F = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}, \quad G = \begin{bmatrix} T^2/2 \\ T \end{bmatrix}$$

with sampling time $T = 1$ and zero-mean white Gaussian process noise w_k having variance $Q = 10$. A simple decentralized fusion system consisting of two nodes is used for tracking this target, but there is no fusion center in the system. The measurements of the two nodes are modeled as

$$z_k^i = H_i x_k + v_k^i, \quad i = 1, 2 \tag{39}$$

where

$$H_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad H_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}$$
$$\text{cov}\left(v_k^i, v_k^i\right) = R_k^{ij},\ i, j = 1, 2,\ i \neq j$$
$$R_k^{ij} = \rho_k \sqrt{R_k^i R_k^j}$$

and ρ_k is the correlation coefficient of the two zero-mean white Gaussian measurement noises at time k.

Each node measures different quantities about the target: node 1 observes position with variance $R_k^1 = 10$; node 2 measures velocity with variance $R_k^2 = 2$. Assume that a full communication rate is employed and the information exchange between nodes is synchronous. Each node applies a Kalman filter to estimate the state. At time k in the ith node, when each node obtains its own measurement, the local estimates can be obtained by using a standard Kalman filter given the previous stored estimate. Then each node propagates its own estimate to another node and fuses its own estimate with the received one to obtain a final fused estimate $\left(\hat{x}_{k|k}^i, P_{k|k}^i\right)$. If both nodes apply the same fusion rule that is symmetric (e.g., AC-CI, GCI), then their final estimates are identical.

It is assumed that both nodes apply the same fusion rules: AC-CI or GCI. The filter is initialized at the true state, which is generated as Gaussian distributed with mean $\bar{x}_0 = \begin{bmatrix} 10 & 5 \end{bmatrix}^T$ and covariance $P_0 =$diag$(100, 25)$.

In Fig. 7, we vary $-1 < \rho < 1$ and compare the RMS position and velocity errors, NCI and I^2 of AC-CI ,GCI and centralized LMMSE fusion (CF) (e.g., a standard Kalman filter with raw data and known correlation R_k^{ij}), at $k = 20\ s$ from 500 Monte Carlo runs. As can be seen, for RMSE, AC-CI is somewhat better than GCI in estimating position for any value of ρ, but GCI is slightly better in estimating velocity, especially when ρ is large; with respect to NCI and I^2, both GCI and AC-CI yields conservative (i.e., pessimistic) estimates. However, since the GCI minimizes the determinant of fused MSE matrix, in general, the MSE matrix provided by GCI may have least uncertainty among CI algorithms. Thus, GCI is more credible and less conservative than AC-CI.

Note that for higher dimensions, the selection of GCI and AC-CI depends on the specific system and problem of interest.

V. Conclusions

This paper has considered the problem of estimation fusion under unknown cross-correlation between the errors of estimates to be fused. We have presented an analytic center (AC-CI) algorithm with respect to a set-theoretic optimization criterion, and obtained an estimate that maximizes the complementary squared Mahalanobis "distance" between the local estimate and the desired estimate in a logarithmic average form. For our problem, we have proven that the analytic center is a convex combination of the local estimates. Thus, the AC-CI algorithm can be viewed as a CI algorithm with respect to a set-theoretic optimization criterion.

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